

# ON EVOLUTION ALGEBRAS

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**ABSTRACT.** The structural constants of an evolution algebra is given by a quadratic matrix  $A$ . In this work we establish equivalence between nil, right nilpotent evolution algebras and evolution algebras, which are defined by upper triangular matrix  $A$ . The classification of 2-dimensional complex evolution algebras is obtained. For an evolution algebra with a special form of the matrix  $A$  we describe all its isomorphisms and their compositions. We construct an algorithm running under Mathematica which decides if two finite dimensional evolution algebras are isomorphic.

**AMS Subject Classifications (2010):** 17D92; 17D99

**Key words:** Evolution algebra, nil algebra, right nilpotent algebra, matrix, group of endomorphisms, classification.

## 1. INTRODUCTION

In this paper we consider a class of algebras called evolution algebras. The concept of evolution algebra lies between algebras and dynamical systems. Algebraically, evolution algebras are non-associative Banach algebras; dynamically, they represent discrete dynamical systems. Evolution algebras have many connections with other mathematical fields including graph theory, group theory, stochastic processes, mathematical physics, etc. [4], [5].

In the book [5] the foundation of evolution algebra theory and applications in non-Mendelian genetics and Markov chains is developed, with pointers to some further research topics.

Let  $(E, \cdot)$  be an algebra over a field  $K$ . If it admits a basis  $e_1, e_2, \dots$ , such that

$$e_i \cdot e_j = 0, \quad \text{if } i \neq j;$$

$$e_i \cdot e_i = \sum_k a_{ik} e_k, \quad \text{for any } i,$$

then this algebra is called an *evolution algebra*. We denote by  $A = (a_{ij})$  the matrix of the structural constants of the evolution algebra  $E$ .

In [2] an evolution algebra  $\mathcal{A}$  associated to the free population is introduced and using this non-associative algebra many results are obtained in explicit form, e.g. the explicit description of stationary quadratic operators, and the explicit solutions of a nonlinear evolutionary equation in the absence of selection, as well as general theorems on convergence to equilibrium in the presence of selection.

In study of any class of algebras, it is important to describe up to isomorphism even algebras of lower dimensions because such description gives examples to establish or reject certain conjectures. In this way in [3] and [6], the classifications of associative and nilpotent Lie algebras of low dimensions were given.

In this paper we study some properties of evolution algebras. The paper is organized as follows. In Section 2 we establish equivalence between nil, right nilpotent evolution algebras and evolution algebras, which are defined by upper triangular matrix  $A$ . In [1] it was proved that these notions are equivalent to the nilpotency of evolution algebras, but right nilindex and nilindex do not coincide in general. Thus it is natural to study conditions when some powers of the evolution algebras are equal to zero. In Section 3 we consider an evolution algebra  $E$  with an upper triangular matrix  $A$  and drive a system of equation (for entries of the matrix  $A$ ) solutions to which gives  $E^k = 0$  for small values of  $k$ . Section 4 is devoted to the classification of 2-dimensional complex evolution algebras. In Section 5 for an evolution algebra with a special form of the matrix  $A$  we describe all its isomorphisms and their compositions. Finally, in Appendix, we construct an algorithm running under Mathematica, using Gröbner bases and the star product of two evolution matrices, which decides if two finite dimensional evolution algebras are isomorphic.

## 2. NIL AND RIGHT NILPOTENT EVOLUTION ALGEBRAS

In this section we prove that notions of nil and right nilpotency are equivalent for evolution algebras. Moreover, the defined matrix  $A$  of such algebras has upper (or lower, up to permutation of basis of the algebra) triangular form.

**Definition 2.1.** *An element  $a$  of an evolution algebra  $E$  is called nil if there exists  $n(a) \in \mathbb{N}$  such that  $(\cdots (\underbrace{(a \cdot a) \cdot a}_{n(a)} \cdots a) = 0$ . Evolution algebra  $E$  is called nil if every element of the algebra is nil.*

**Theorem 2.2.** *Let  $E$  be a nil evolution algebra with basis  $\{e_1, \dots, e_n\}$ . Then for the elements of the matrix  $A = (a_{ij})$  the following relation*

holds

$$(2.1) \quad a_{i_1 i_2} a_{i_2 i_3} \dots a_{i_k i_1} = 0,$$

for all  $i_1, \dots, i_k \in \{1, \dots, n\}$  and  $k \in \{1, \dots, n\}$ , with  $i_p \neq i_q$  for  $p \neq q$ .

*Proof.* Note that  $((e_i \cdot e_i) \cdot e_i) = a_{ii} e_i^2$ , hence  $a_{ii} = 0$  (otherwise the element  $e_i$  is not nil). We shall prove the equality (2.1) for right normed terms by induction.

For the element  $e_i + e_j$ ,  $1 \leq i, j \leq n$ , it can be proved by induction the following relation

$$(e_i + e_j)^{2s} = a_{ij}^{s-1} a_{ji}^{s-1} (e_i + e_j)^2.$$

The nil condition for the element  $e_i + e_j$  leads to the equality for the elements of the matrix  $A$ :

$$a_{ij} a_{ji} = 0 \text{ or } e_i^2 + e_j^2 = 0.$$

Take in account the fact that  $a_{ii} = a_{jj} = 0$  for any  $i$  and  $j$  and comparing the coefficients at the basic elements, from condition  $e_i^2 + e_j^2 = 0$  we obtain  $a_{ij} = a_{ji} = 0$ . Hence, the equation  $a_{ij} a_{ji} = 0$  for all  $i, j$  is obtained and therefore the equality (2.1) is true for  $k = 2$ .

Let (2.1) be true for  $k-1$ . We shall prove it for  $k$ . For this purpose we consider element  $e_{i_1} + e_{i_2} + \dots + e_{i_k}$ . Without loss of generality instead of this element we can consider the following element  $e_1 + e_2 + \dots + e_k$ . Using the hypothesis of the induction it is not difficult to note that

$$\left( \sum_{i=1}^k e_i \right)^{s+2} = \sum_{\substack{i_1, \dots, i_s=1 \\ i_p \neq i_q, p \neq q}}^k a_{i_1 i_2} a_{i_2 i_3} a_{i_3 i_4} \dots a_{i_{s-1} i_s} e_{i_s}^2.$$

Let us take in the above expression  $s = k+1$ , then  $i_{s-1} = i_k$ . From induction hypothesis the coefficient  $a_{i_1 i_2} a_{i_2 i_3} a_{i_3 i_4} \dots a_{i_{s-1} i_s}$  is equal to zero if  $i_s \in \{i_2, \dots, i_{s-1}\}$ . Therefore we need to consider the case  $i_s = i_1$  and the above expression will have the following form

$$\begin{aligned} \left( \sum_{i=1}^k e_i \right)^{k+3} &= \sum_{\phi \in S_k} a_{\phi(1)\phi(2)} a_{\phi(2)\phi(3)} \dots a_{\phi(k)\phi(1)} e_{\phi(1)}^2 = \\ &= \sum_{i=1}^k \left( \sum_{\phi \in S_k: \phi(1)=i} a_{i\phi(2)} a_{\phi(2)\phi(3)} \dots a_{\phi(k)i} \right) e_i^2, \end{aligned}$$

where  $S_k$  denotes the symmetric group of permutations of  $k$  elements.

Denote

$$\mathcal{F}_i = \sum_{\phi \in S_k: \phi(1)=i} a_{i\phi(2)} a_{\phi(2)\phi(3)} \dots a_{\phi(k)i}.$$

We need the following lemmas

**Lemma 2.3.** *For any  $i, j = 1, \dots, k$  we have  $\mathcal{F}_i = \mathcal{F}_j$ .*

*Proof.* For  $\phi \in S_k$  with  $\phi(1) = i$  we construct a unique  $\bar{\phi} \in S_k$  such that  $\bar{\phi}(1) = j$  and

$$(2.2) \quad a_{i\phi(2)}a_{\phi(2)\phi(3)} \dots a_{\phi(k)i} = a_{j\bar{\phi}(2)}a_{\bar{\phi}(2)\bar{\phi}(3)} \dots a_{\bar{\phi}(k)j}$$

as follows. Let  $s$  be the number such that  $\phi(s) = j$ , then the permutation  $\bar{\phi}$  is defined as

$$\bar{\phi} = \begin{pmatrix} 1 & 2 & \dots & k-s+1 & k-s+2 & k-s+3 & \dots & k \\ j & \phi(s+1) & \dots & \phi(k) & i & \phi(2) & \dots & \phi(s-1) \end{pmatrix}.$$

By construction, we note that for a given  $\phi$  the  $\bar{\phi}$  is uniquely defined and satisfies (2.2). Thus we get  $\mathcal{F}_i = \mathcal{F}_j$ .  $\square$

Put  $a = \sum_{i=1}^k e_i$ .

**Lemma 2.4.** *If  $a^2 = 0$  then  $\mathcal{F}_1 = 0$ .*

*Proof.* From  $a^2 = 0$  we obtain

$$\sum_{\substack{i=1 \\ i \neq j}}^k a_{ij} = 0, \quad j = 1, \dots, n.$$

Using this equality we get

$$\begin{aligned} \mathcal{F}_1 &= \sum_{\phi \in S_k: \phi(1)=1} a_{1\phi(2)}a_{\phi(2)\phi(3)} \dots a_{\phi(k)1} = \\ &= \sum_{\phi \in S_k: \phi(1)=1} \sum_{\substack{i=2 \\ i \neq \phi(2)}}^k a_{i\phi(2)}a_{\phi(2)\phi(3)} \dots a_{\phi(k)1}. \end{aligned}$$

Since for any  $i = 2, \dots, k$  there exists  $s_i$  such that  $\phi(s_i) = i$ , by the assumption of the induction we get

$$a_{i\phi(2)}a_{\phi(2)\phi(3)} \dots a_{\phi(k)1} = a_{i\phi(2)}a_{\phi(2)\phi(3)} \dots a_{\phi(s_i-1)i}a_{i\phi(s_i+1)} \dots a_{\phi(k)1} = 0.$$

$\square$

Now we continue the proof of theorem. Using Lemma 2.3, we get

$$\left( \sum_{i=1}^k e_i \right)^{k+3} = \mathcal{F}_1 a^2 = 0.$$

By Lemma 2.4 we get  $\mathcal{F}_1 = 0$ . Fix an arbitrary  $\phi_0 \in S_k$  with  $\phi_0(1) = 1$  and multiply both side of  $\mathcal{F}_1 = 0$  by  $a_{1\phi_0(2)}a_{\phi_0(2)\phi_0(3)} \cdots a_{\phi_0(k)1}$  then (again using the assumption of the induction) we obtain

$$a_{1\phi_0(2)}^2 a_{\phi_0(2)\phi_0(3)}^2 \cdots a_{\phi_0(k)1}^2 = 0,$$

i.e.

$$a_{1\phi_0(2)}a_{\phi_0(2)\phi_0(3)} \cdots a_{\phi_0(k)1} = 0,$$

which completes the induction and the proof of theorem.  $\square$

For an evolution algebra  $E$  we introduce the following sequence

$$E^{<1>} = E, \quad E^{<k+1>} = E^{<k>}E, \quad k \geq 1.$$

**Definition 2.5.** *An evolution algebra is called right nilpotent if there exists some  $s \in \mathbb{N}$  such that  $E^{<s>} = 0$ .*

Let evolution algebra  $E$  be a right nilpotent algebra, then it is evident that  $E$  is nil algebra. Therefore for the related matrix  $A = (a_{ij})_{i,j=1}^n$  we have

$$(2.3) \quad a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_k i_1} = 0,$$

for any  $k \in \{1, 2, \dots, n\}$  and arbitrary  $i_1, i_2, \dots, i_k \in \{1, 2, \dots, n\}$  with  $i_p \neq i_q$  for  $p \neq q$ .

**Lemma 2.6.** *Let the matrix  $A$  satisfies (2.3). Then for any  $j \in \{1, \dots, n\}$  there is a row  $\pi_j$  of  $A$  with  $j$  zeros. Moreover,  $\pi_{j_1} \neq \pi_{j_2}$  if  $j_1 \neq j_2$ .*

*Proof.* First we shall prove that there is a row with  $j = n$  zeros, i.e. all zeros. Assume that there is not such a row. Then for any  $i \in \{1, \dots, n\}$  there is a number  $\beta(i) \in \{1, \dots, n\} \setminus \{i\}$  such that  $a_{i\beta(i)} \neq 0$ . Consider the sequence

$$i_1 = 1, i_2 = \beta(1), i_3 = \beta(\beta(1)), \dots, i_{n+1} = \underbrace{\beta(\cdots(\beta(1)))}_n.$$

Then by assumption we have  $a_{i_m i_{m+1}} \neq 0$ , for all  $m = 1, \dots, n$  hence

$$(2.4) \quad a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_n i_{n+1}} \neq 0.$$

Since  $i_j \in \{1, \dots, n\}$ ,  $j = 1, \dots, n+1$  between them there are  $i_p = i_q$  for some  $p \neq q \in \{1, \dots, n+1\}$ . Thus

$$(2.5) \quad a_{i_p i_{p+1}} a_{i_{p+1} i_{p+2}} \cdots a_{i_{q-1} i_q} \neq 0.$$

So (2.5) is in contradiction with (2.3). Thus there is a row  $\pi_n$  with all zeros ( $n$  zeros).

Now we shall prove that there is a row  $\pi_{n-1} \neq \pi_n$  of  $A$  with  $n-1$  zeros. Consider  $A_{\pi_n}$ -minor of  $A$  which is constructed by  $A$  deleting row

$\pi_n$  and column  $\pi_n$ . Matrix  $A_{\pi_n}$  is  $(n-1) \times (n-1)$  and condition (2.3) implies the condition

$$(2.6) \quad a_{i_1 i_2} a_{i_2 i_3} \dots a_{i_k i_1} = 0,$$

for any  $k \in \{1, \dots, n-1\}$  and arbitrary  $i_1, i_2, \dots, i_k \in \{1, \dots, n\} \setminus \{\pi_n\}$  with  $i_p \neq i_q$  for all  $p \neq q$ .

To prove that  $A$  has a row with  $j = n-1$  zeros it is enough to prove that  $A_{\pi_n}$  has a row with all zeros. But this problem is the same as the case  $j = n$ , only we must consider condition (2.6) instead of (2.3). Iterating this argument we can show that for any  $j$  there exists a row  $\pi_j$  with  $j$  zeros. The proof of the lemma is completed.  $\square$

The following theorem is the main result of this section.

**Theorem 2.7.** *The following statements are equivalent for an  $n$ -dimensional evolution algebra  $E$ :*

a) *The matrix corresponding to  $E$  can be written as*

$$(2.7) \quad \hat{A} = \begin{pmatrix} 0 & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & 0 & a_{23} & \dots & a_{2n} \\ 0 & 0 & 0 & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix};$$

b)  *$E$  is right nilpotent algebra;*

c)  *$E$  is nil algebra.*

*Proof.* **b)  $\Rightarrow$  a).** Since the equality (2.3) is true for right nilpotent algebra then we are in the conditions of the Lemma 2.6. Consider the permutation of the first indexes  $\{1, \dots, n\}$  of the matrix  $A$  as

$$\pi = \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ \pi_1 & \pi_2 & \pi_3 & \dots & \pi_n \end{pmatrix},$$

where  $\pi_j$  is defined in the proof of the Lemma 2.6. Note that Lemma 2.6 is also true for columns: for any  $j$  there is a column  $\tau_j$  with  $j$  zeros. Moreover  $\tau_p \neq \tau_q$ ,  $p \neq q$ . Now consider permutation of the second indexes  $\{1, \dots, n\}$  as

$$\tau = \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ \tau_1 & \tau_2 & \tau_3 & \dots & \tau_n \end{pmatrix}.$$

Then  $\tau(\pi(A)) = \hat{A}$ .

The implication **b)**  $\Rightarrow$  **c)** is evident since every right nilpotent evolution algebra is nil algebra.

The implication **a)**  $\Rightarrow$  **b), c)** is also true, because the table of the multiplication of the evolution algebra defined by upper triangular matrix  $A$  will be right nilpotent and nil.

The implication **c)**  $\Rightarrow$  **a)** follows from Theorem 2.2.  $\square$

### 3. CONDITIONS FOR $E^k = 0$

For an evolution algebra  $E$  we define the *lower central sequence* by:

$$E^1 = E, \quad E^k = \sum_{i=1}^{k-1} E^i E^{k-i}, \quad k \geq 1.$$

An evolution algebra  $E$  is called *nilpotent* if there exists  $n \in \mathbb{N}$  such that  $E^n = 0$ . In [1], it is proved that the notions of nilpotent and right nilpotent are equivalent.

In this section we consider an  $n$ -dimensional evolution algebra  $E$  with a triangular (as  $\hat{A}$  in Theorem 2.7) matrix  $A$  and for small values of  $k$  we present conditions on entries of  $A$  under which  $E^k = 0$ .

First, for  $n = 3$  we have  $E^2 = 0 \Leftrightarrow a_{ij} \equiv 0$  and  $E^3 = 0 \Leftrightarrow a_{12}a_{23} = 0$ . For  $n = 4$  one easily finds  $E^2 = 0 \Leftrightarrow a_{ij} \equiv 0$  and

$$E^3 = 0 \Leftrightarrow a_{12}a_{23} = 0, a_{12}a_{24} = 0, a_{13}a_{34} = 0, a_{23}a_{34} = 0.$$

Now we consider an arbitrary  $n \in \mathbb{N}$  and for  $k = 3, 4, 5$ , we shall drive solutions of a system of equations (for  $a_{ij}$ ) which give  $E^k = 0, k = 3, 4, 5$ .

Let  $E = \langle e_1, \dots, e_n \rangle$  be an evolution algebra with matrix (2.7).

*Case  $k = 3$ :* We have  $e_i^2 e_j = e_j e_i^2, \forall i, j; e_i e_j e_k = 0, i \neq j$ . Thus we get

$$e_i^2 e_j = \begin{cases} 0 & \text{if } j \leq i; \\ \sum_{s=j+1}^n a_{ij} a_{js} e_s & \text{if } j \geq i+1, s \geq j+1. \end{cases}$$

So the system of equations is

$$a_{ij} a_{js} = 0 \quad \text{for } j \geq i+1, s \geq j+1, i, j = 1, \dots, n. \quad (n; 3)$$

*Case  $k = 4$ :* Since  $e_i^2 e_j^2 = e_j^2 e_i^2; (e_i^2 e_j) e_s = (e_j e_i^2) e_s = e_s (e_j e_i^2) = e_s (e_i^2 e_j); e_i e_j e_s e_t = 0$ , if  $i \neq j; (e_i e_j e_s) e_t = e_t (e_i e_j e_s), \dots$  it will be enough to consider  $e_i^2 e_j^2$  and  $(e_i^2 e_j) e_s$ . We have

$$e_i^2 e_j^2 = \sum_{t=j+2}^n \left( \sum_{u=j+1}^{t-1} a_{iu} a_{ju} a_{ut} \right) e_t, \quad i \leq j, i, j = 1, \dots, n;$$

$$(e_i^2 e_j) e_s = \sum_{t=s+1}^n a_{ij} a_{js} a_{st} e_t, \quad j \geq i+1, \quad s \geq j+1.$$

So the system of equations is

$$\begin{cases} \sum_{u=j+1}^{t-1} a_{iu} a_{ju} a_{ut} = 0 & \text{if } j \leq i, t \geq j+2, i, j = 1, \dots, n; \\ a_{ij} a_{js} a_{st} = 0 & \text{if } j \geq i+1, s \geq j+1, t \geq s+1. \end{cases} \quad (n; 4)$$

Case  $k = 5$ : We should only use previous non-zero words and multiply them to get a word of length 5:

$$\begin{aligned} e_i^2 e_j^2 e_s &= \sum_{t=s+1}^n \left( \sum_{u=j+1}^{s-1} a_{iu} a_{ju} a_{us} a_{st} \right) e_t, \quad i \leq j, s \geq j+2, \\ (e_i^2 e_j) e_s^2 &= a_{ij} \sum_{u=s+2}^n \left( \sum_{t=s+1}^{u-1} a_{jt} a_{st} a_{tu} \right) e_u, \quad j \leq i+1, s \geq j, \\ e_i^2 e_j e_s e_v &= \sum_{u=v+1}^n (a_{ij} a_{js} a_{sv} a_{vu}) e_u, \quad j \geq i+1, s \geq j+1, v \geq s+1. \end{aligned}$$

Thus we get the following system of equations

$$\begin{cases} \sum_{u=j+1}^{s-1} a_{iu} a_{ju} a_{us} a_{st} = 0 & \text{if } j \leq i, s \geq j+2, t \geq s+1, \\ a_{ij} \sum_{t=s+1}^{u-1} a_{jt} a_{st} a_{tu} = 0 & \text{if } j \geq i+1, s \geq j, u \geq s+2, \\ a_{ij} a_{js} a_{sv} a_{vu} = 0. \end{cases} \quad (n; 5)$$

Thus we have proved the following

**Theorem 3.1.** *Let  $E$  be an evolution algebra with matrix (2.7) then  $E^k = 0$  if the elements of the matrix (2.7) satisfy the equations  $(n; k)$ , where  $k = 3, 4, 5$ .*

#### 4. CLASSIFICATION OF COMPLEX 2-DIMENSIONAL EVOLUTION ALGEBRAS

In this section we give the classification of 2-dimensional complex evolution algebras.

Let  $E$  and  $E'$  be evolution algebras and  $\{e_i\}$  a natural basis of  $E$ . A linear map  $\varphi: E \rightarrow E'$  is called an *homomorphism* of evolution algebras if it is an algebraic map and if the set  $\{\varphi(e_i)\}$  can be complemented to a natural basis of  $E'$ . Moreover, if  $\varphi$  is bijective, then it is called an *isomorphism*.

Let  $E$  be a 2-dimensional complex evolution algebra and  $\{e_1, e_2\}$  be a basis of the algebra  $E$ .

It is evident that if  $\dim E^2 = 0$  then  $E$  is an abelian algebra, i.e. an algebra with all products zero.



**Theorem 4.1.** *Any 2-dimensional complex evolution algebra  $E$  is isomorphic to one of the following pairwise non isomorphic algebras:*

(1)  $\dim E^2 = 1$

- $E_1$  :  $e_1e_1 = e_1$ ,
- $E_2$  :  $e_1e_1 = e_1, \quad e_2e_2 = e_1$ ,
- $E_3$  :  $e_1e_1 = e_1 + e_2, \quad e_2e_2 = -e_1 - e_2$ ,
- $E_4$  :  $e_1e_1 = e_2$ .

(2)  $\dim E^2 = 2$

- $E_5$  :  $e_1e_1 = e_1 + a_2e_2, \quad e_2e_2 = a_3e_1 + e_2, \quad 1 - a_2a_3 \neq 0$ ,  
where  $E_5(a_2, a_3) \cong E'_5(a_3, a_2)$ ,
- $E_6$  :  $e_1e_1 = e_2, \quad e_2e_2 = e_1 + a_4e_2, \quad a_4 \neq 0$ , where  $E_6(a_4) \cong E_6(a'_4) \Leftrightarrow \frac{a'_4}{a_4} = \cos \frac{2\pi k}{3} + i \sin \frac{2\pi k}{3}$  for some  $k = 0, 1, 2$ .

*Proof.* For an evolution algebra  $E$  we have

$$e_1e_1 = a_1e_1 + a_2e_2, \quad e_2e_2 = a_3e_1 + a_4e_2, \quad e_1e_2 = e_2e_1 = 0.$$

Since  $\dim E^2 = 1$ , then

$$e_1e_1 = c_1(a_1e_1 + a_2e_2), \quad e_2e_2 = c_2(a_1e_1 + a_2e_2), \quad e_1e_2 = e_2e_1 = 0.$$

Evidently  $(c_1, c_2) \neq (0, 0)$ , because otherwise our algebra will be abelian.

Since  $e_1$  and  $e_2$  are symmetric we can suppose that  $c_1 \neq 0$ , then by simple change of basis (scale of it) we can do  $c_1 = 1$ .

**Case 1.**  $a_1 \neq 0$ . Then we take the following change of basis

$$e'_1 = a_1e_1 + a_2e_2, \quad e'_2 = Ae_1 + Be_2,$$

where  $a_1B - a_2A \neq 0$ .

Consider the product

$$\begin{aligned} 0 &= e'_1e'_2 = (a_1e_1 + a_2e_2)(Ae_1 + Be_2) = a_1A(a_1e_1 + a_2e_2) + \\ &\quad a_2Bc_2(a_1e_1 + a_2e_2) = (a_1A + a_2Bc_2)(a_1e_1 + a_2e_2). \end{aligned}$$

Therefore,  $a_1A + a_2Bc_2 = 0$ , i.e.  $A = -\frac{a_2Bc_2}{a_1}$  and  $a_1B - a_2A = a_1B + \frac{a_2^2Bc_2}{a_1} \neq 0$ .

It means that in the case when  $a_1^2 + a_2^2c_2 \neq 0$  we can take the above change.

Consider the products

$$\begin{aligned} e'_1e'_1 &= (a_1e_1 + a_2e_2)(a_1e_1 + a_2e_2) = a_1^2(a_1e_1 + a_2e_2) + a_2^2c_2(a_1e_1 + a_2e_2) = \\ &\quad (a_1^2 + a_2^2c_2)(a_1e_1 + a_2e_2) = (a_1^2 + a_2^2c_2)e'_1, \\ e'_2e'_2 &= (Ae_1 + Be_2)(Ae_1 + Be_2) = A^2(a_1e_1 + a_2e_2) + B_2^2c_2(a_1e_1 + a_2e_2) = \end{aligned}$$

$$(A^2 + B^2 c_2)(a_1 e_1 + a_2 e_2) = \left( \frac{a_2^2 B^2 c_2^2}{a_1^2} + B^2 c_2 \right) e'_1 = \frac{B^2 c_2 (a_1^2 + a_2^2 c_2)}{a_1^2} e'_1.$$

**Case 1.1.:**  $c_2 = 0$ . Then  $e_1 e_1 = a_1^2 e_1$ ,  $e_2 e_2 = e_1 e_2 = e_2 e_1 = 0$ . Taking  $e'_1 = \frac{e_1}{a_1^2}$  we get the algebra  $E_1$ .

**Case 1.2.:**  $c_2 \neq 0$ . Then taking  $B = \sqrt{\frac{a_1^2}{c_2}}$  we obtain

$$e_1 e_1 = (a_1^2 + a_2^2 c_2) e_1, \quad e_2 e_2 = (a_1^2 + a_2^2 c_2) e_1.$$

If  $a_1^2 + a_2^2 c_2 \neq 0$ , the following change of basis

$$e'_1 = \frac{e_1}{a_1^2 + a_2^2 c_2}, \quad e'_2 = \frac{e_2}{a_1^2 + a_2^2 c_2}$$

derives to the algebra with multiplication:

$$e_1 e_1 = e_1, \quad e_2 e_2 = e_1.$$

If  $a_1^2 + a_2^2 c_2 = 0$ , then  $c_2 = -\frac{a_1^2}{a_2^2}$  and we have  $e_1 e_1 = a_1 e_1 + a_2 e_2$ ,  $e_2 e_2 = -\frac{a_1^3}{a_2^2} e_1 - \frac{a_1^2}{a_2} e_2$ .

The change of basis  $e'_1 = \frac{e_1}{a_1}$ ,  $e'_2 = \frac{a_2}{a_1^2} e_2$  derives to the algebra  $E_3$ .

**Case 2.**  $a_1 = 0$ . Then we have the products  $e_1 e_1 = a_2 e_2$ ,  $e_2 e_2 = c_2 a_2 e_2$ , where  $a_2 \neq 0$ .

If  $c_2 = 0$ , then by the change  $e'_1 = \frac{e_1}{\sqrt{a_2}}$  we get again the algebra  $E_4$ .

If  $c_2 \neq 0$ , then by  $e'_1 = \frac{e_1}{\sqrt{c_2 a_2^2}}$ ,  $e'_2 = \frac{e_2}{c_2 a_2}$  we get the algebra  $e_1 e_1 = e_2$ ,  $e_2 e_2 = e_2$  which is isomorphic to the algebra  $E_2$ .

Now we consider algebras with  $\dim E^2 = 2$ . Let us write the table of multiplication:

$$e_1 e_1 = a_1 e_1 + a_2 e_2, \quad e_2 e_2 = a_3 e_1 + a_4 e_2,$$

where  $a_1 a_4 - a_2 a_3 \neq 0$ .

**Case 1.**  $a_1 \neq 0$  and  $a_4 \neq 0$ . Then we can transform both of them to unit, i.e. we can suppose  $a_1 = a_4 = 1$ . Therefore we have the two parametric family  $E_5(a_2, a_3)$ :

$$e_1 e_1 = e_1 + a_2 e_2, \quad e_2 e_2 = a_3 e_1 + e_2, \quad 1 - a_2 a_3 \neq 0.$$

Let us take the general change of basis of the form

$$e'_1 = A_1 e_1 + A_2 e_2, \quad e'_2 = B_1 e_1 + B_2 e_2,$$

where  $A_1 B_2 - A_2 B_1 \neq 0$ .

Consider the product

$$0 = e'_1 e'_2 = (A_1 e_1 + A_2 e_2)(B_1 e_1 + B_2 e_2) = A_1 B_1 (e_1 + a_2 e_2) + A_2 B_2 (a_3 e_1 + e_2) = (A_1 B_1 + A_2 B_2 a_3) e_1 + (A_1 B_1 a_2 + A_2 B_2) e_2.$$

Since in this new basis the algebra should be also evolution we have

$$A_1B_1 + A_2B_2a_3 = 0, \quad A_1B_1a_2 + A_2B_2 = 0.$$

From which we have  $A_2B_2(1 - a_2a_3) = 0$ ,  $A_1B_1(1 - a_2a_3) = 0$ . Since  $1 - a_2a_3 \neq 0$ , then we have  $A_1B_1 = A_2B_2 = 0$ .

**Case 1.1.:**  $A_2 = 0$ . Then  $B_1 = 0$ .

Consider the products

$$\begin{aligned} e'_1e'_1 &= A_1^2(e_1 + a_2e_2) = e'_1 + a'_2e'_2 = A_1e_1 + a'_2B_2e_2 \Rightarrow A_1^2 = A_1, \quad A_1^2a_2 = a'_2B_2 \Rightarrow A_1 = 1, \\ e'_2e'_2 &= B_2^2(a_3e_1 + e_2) = a'_3e'_1 + e'_2 = a'_3A_1e_1 + B_2e_2 \Rightarrow B_2^2a_3 = a'_3A_1, \quad B_2^2 = B_2 \Rightarrow B_2 = 1. \end{aligned}$$

**Case 1.2.:**  $A_1 = 0$ . Then  $B_2 = 0$  and from the family of algebras  $E_5(a_2, a_3)$  we get the family  $E_5(a_3, a_2)$ .

**Case 2.**  $a_1 = 0$  or  $a_4 = 0$ . Since  $e_1$  and  $e_2$  are symmetric then without loss of generality we can suppose that  $a_1 = 0$ .

$$e_1e_1 = a_2e_2, \quad e_2e_2 = a_3e_1 + a_4e_2,$$

where  $a_2a_3 \neq 0$ .

Taking the change of basis  $e'_1 = \sqrt[3]{\frac{1}{a_2^2a_3}}e_1$ ,  $e'_2 = \sqrt[3]{\frac{1}{a_2a_3^2}}e_2$  we obtain the one-parametric family of algebras  $E_6(a_4)$ :

$$e_1e_1 = e_2, \quad e_2e_2 = e_1 + a_4e_2.$$

Let us take the general change of basis

$$e'_1 = A_1e_1 + A_2e_2, \quad e'_2 = B_1e_1 + B_2e_2,$$

where  $A_1B_2 - A_2B_1 \neq 0$ .

Consider the product

$$0 = e'_1e'_2 = (A_1e_1 + A_2e_2)(B_1e_1 + B_2e_2) = A_1B_1e_2 + A_2B_2(e_1 + a_4e_2).$$

Therefore  $A_1B_1 + A_2B_2a_4 = 0$ ,  $A_2B_2 = 0 \Rightarrow A_1B_1 = 0$ ,  $A_2B_2 = 0$ .

Without loss of generality we can assume that  $A_2 = 0$ . Then  $B_1 = 0$ .

Consider the product

$$\begin{aligned} e'_1e'_1 &= A_1^2e_2 = e'_2 = B_2e_2 \Rightarrow A_1^2 = B_2, \\ e'_2e'_2 &= B_2^2(e_1 + a_4e_2) = e'_1 + a'_4e'_2 = A_1e_1 + a'_4B_2e_2 \Rightarrow B_2^2 = A_1, \quad B_2^2a_4 = B_2a'_4. \end{aligned}$$

From these equalities we have  $B_2^3 = 1$ ,  $B_2a_4 = a'_4$ .

If  $\frac{a'_4}{a_4} = \cos \frac{2\pi k}{3} + i \sin \frac{2\pi k}{3}$  for some  $k = 0, 1, 2$ , then putting  $B_2 = \cos \frac{2\pi k}{3} + i \sin \frac{2\pi k}{3}$  we obtain the isomorphism between algebras  $E_6(a_4)$  and  $E_6(a'_4)$ .

The pairwise non isomorphic obtained algebras can be checked by comparison of the algebraic properties listed in the following table.

	$\dim E^2$	Right Nilpotency	$\dim(\text{Annihilator})$	Nil Elements
$E_1$	1	No	1	Yes
$E_2$	1	No	0	Yes
$E_3$	1	No	0	No
$E_4$	1	Yes	1	Yes
$E_5$	2	No	0	Non
$E_6$	2	No	0	Yes

□

### 5. ISOMORPHISMS OF EVOLUTION ALGEBRAS

Since the study of the isomorphisms for any class of algebras is a crucial task and taking into account the great difficulties of their description, in this section we consider a particular case of evolution algebras, which have matrices in the diagonal  $2 \times 2$  non-zero blocks.

Let  $E$  be an evolution algebra which has a matrix  $A$  in the following form

$$A = \begin{pmatrix} a_1 & b_1 & 0 & 0 & \dots & 0 & 0 \\ c_1 & d_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & a_2 & b_2 & \dots & 0 & 0 \\ 0 & 0 & c_2 & d_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & a_n & b_n \\ 0 & 0 & 0 & 0 & \dots & c_n & d_n \end{pmatrix}.$$

The basis  $\{e_1, \dots, e_{2n}\}$  of this evolution algebra has the following relations:

$$\begin{aligned} e_i e_j &= 0, i \neq j; \quad e_{2k-1}^2 = a_k e_{2k-1} + b_k e_{2k}, \quad k = 1, 2, \dots, n; \\ e_{2k}^2 &= c_k e_{2k-1} + d_k e_{2k}, \quad k = 1, 2, \dots, n. \end{aligned}$$

Let  $\varphi$  be an isomorphism of the evolution algebra  $E$  onto  $E$  with matrix  $A'$ . Write  $\varphi$  as

$$\varphi = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1,2n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2,2n} \\ \vdots & \vdots & \dots & \vdots \\ \alpha_{2n,1} & \alpha_{2n,2} & \dots & \alpha_{2n,2n} \end{pmatrix},$$

with  $\det(\varphi) \neq 0$ . We have

$$(e'_i)^2 = (\varphi(e_i))^2 = (\alpha_{i1}^2 a_1 + \alpha_{i2}^2 c_1) e_1 + (\alpha_{i1}^2 b_1 + \alpha_{i2}^2 d_1) e_2 +$$

$$+ \cdots + (\alpha_{i,2n-1}^2 a_n + \alpha_{i,2n}^2 c_n) e_{2n-1} + (\alpha_{i,2n-1}^2 b_n + \alpha_{i,2n}^2 d_n) e_{2n}; \quad i = 1, 2, \dots, 2n.$$

For  $i \neq j$  we get

$$(5.1) \quad \begin{aligned} e'_i e'_j &= (\alpha_{i1} \alpha_{j1} a_1 + \alpha_{i2} \alpha_{j2} c_1) e_1 + (\alpha_{i1} \alpha_{j1} b_1 + \alpha_{i2} \alpha_{j2} d_1) e_2 + \cdots \\ &+ (\alpha_{i,2n-1} \alpha_{j,2n-1} a_n + \alpha_{i,2n} \alpha_{j,2n} c_n) e_{2n-1} + (\alpha_{i,2n-1} \alpha_{j,2n-1} b_n + \alpha_{i,2n} \alpha_{j,2n} d_n) e_{2n} = 0. \end{aligned}$$

From (5.1) we obtain

$$(5.2) \quad \begin{cases} \alpha_{i1} \alpha_{j1} a_1 + \alpha_{i2} \alpha_{j2} c_1 = 0 \\ \alpha_{i1} \alpha_{j1} b_1 + \alpha_{i2} \alpha_{j2} d_1 = 0 \\ \dots \\ \alpha_{i,2n-1} \alpha_{j,2n-1} a_n + \alpha_{i,2n} \alpha_{j,2n} c_n = 0 \\ \alpha_{i,2n-1} \alpha_{j,2n-1} b_n + \alpha_{i,2n} \alpha_{j,2n} d_n = 0. \end{cases}$$

Let  $S_{2n}$  be the group of permutations of  $1, 2, \dots, 2n$ .

**Theorem 5.1.** *Assume that  $\det(A) \neq 0$  then*

(i) *For any isomorphism  $\varphi : E \rightarrow E$  there exists unique  $\pi = \pi(\varphi) \in S_{2n}$  such that*

$$\varphi \in \Phi_\pi = \left\{ \begin{pmatrix} \alpha_{11} & \dots & \alpha_{1,2n} \\ \alpha_{21} & \dots & \alpha_{2,2n} \\ \vdots & \dots & \vdots \\ \alpha_{2n,1} & \dots & \alpha_{2n,2n} \end{pmatrix} : \begin{array}{l} \alpha_{i\pi(i)} \neq 0, \quad 1 \leq i \leq 2n \\ \text{and the rest of elements } \alpha_{ij} = 0 \end{array} \right\}.$$

Moreover, the set  $\Phi = \cup_{\pi \in S_{2n}} \Phi_\pi$  is the set of all possible homomorphisms.

(ii) *For any  $\pi, \tau \in S_{2n}$  the following equality holds*

$$\Phi_\pi \Phi_\tau = \{\varphi\psi : \varphi \in \Phi_\pi, \psi \in \Phi_\tau\} = \Phi_{\tau\pi}.$$

The set  $G = \{\Phi_\pi : \pi \in S_{2n}\}$  is a multiplicative group.

*Proof.* (i) Since  $\det(A) \neq 0$  we have  $a_i d_i - b_i c_i \neq 0$  for any  $i = 1, 2, \dots, n$ . Thus from (5.2) we have

$$(5.3) \quad \alpha_{ik} \alpha_{jk} = 0, \quad i \neq j, \quad i, j, k = 1, \dots, 2n.$$

By (5.3) it is easy to see that each row and each column of the matrix  $\varphi$  must contain exactly one non-zero element. It is not difficult to see that every such matrix  $\varphi$  corresponds to a permutation  $\pi$ . The set of all possible solutions of (5.3) give all the possible isomorphisms, i.e. we get the set  $\Phi$ .

(ii) Take  $\varphi = \{\alpha_{ij}\} \in \Phi_\pi$  and  $\psi = \{\beta_{ij}\} \in \Phi_\tau$ . Denote  $\varphi \circ \psi = \{\gamma_{ij}\}$ . It is easy to see that

$$\gamma_{ij} = \begin{cases} 0 & \text{if } j \neq \tau(\pi(i)); \\ \alpha_{i\pi(i)}\beta_{\pi(i)\tau(\pi(i))} & \text{if } j = \tau(\pi(i)). \end{cases}$$

This gives  $\Phi_\pi\Phi_\tau = \Phi_{\tau\pi}$  and then one easily can check that  $G$  is a group.  $\square$

Now for a fixed  $\varphi$  (i.e.  $\pi$ ) we shall find the matrix  $A'$ . Consider  $\pi \in S_{2n}$  and the corresponding  $\varphi_\pi = (\alpha_{ij})$ :

$$\alpha_{ij} = \begin{cases} 0 & \text{if } j \neq \pi(i); \\ \alpha_{i\pi(i)} & \text{if } j = \pi(i). \end{cases}$$

We have

$$(5.4) \quad e'_i = \alpha_{i\pi(i)}e_{\pi(i)}, \quad i = 1, \dots, 2n.$$

Using this equality we get

$$(e'_i)^2 = \alpha_{i\pi(i)}^2 e_{\pi(i)}^2 = \begin{cases} \alpha_{i(2k-1)}^2 (a_k e_{2k-1} + b_k e_{2k}), & \text{if } \pi(i) = 2k-1; \\ \alpha_{i(2k)}^2 (c_k e_{2k-1} + d_k e_{2k}), & \text{if } \pi(i) = 2k. \end{cases}$$

By (5.4) from the last equality we get

$$(e'_i)^2 = \begin{cases} (\alpha_{i(2k-1)}a_k)e'_i + \left(\frac{\alpha_{i(2k-1)}^2}{\alpha_{i(2k)}}b_k\right)e'_{\pi^{-1}(2k)}, & \text{if } \pi(i) = 2k-1; \\ \left(\frac{\alpha_{i(2k)}^2}{\alpha_{i(2k-1)}}c_k\right)e'_{\pi^{-1}(2k-1)} + (\alpha_{i(2k)}d_k)e'_i, & \text{if } \pi(i) = 2k. \end{cases}$$

Thus  $A' = (a'_{ij})$  is a matrix with

$$(5.5) \quad a'_{ij} = \begin{cases} \alpha_{i(2k-1)}a_k & \text{if } \pi(i) = 2k-1, j = i; \\ \frac{\alpha_{i(2k-1)}^2}{\alpha_{i(2k)}}b_k, & \text{if } \pi(i) = 2k-1, \pi(j) = 2k; \\ \frac{\alpha_{i(2k)}^2}{\alpha_{i(2k-1)}}c_k & \text{if } \pi(i) = 2k, \pi(j) = 2k-1; \\ \alpha_{i(2k)}d_k, & \text{if } \pi(i) = 2k, j = i; \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 5.2.** *Assume that  $\det(A) \neq 0$ . Let  $\varphi : E \rightarrow E$  ( $A \rightarrow A'$ ) be an isomorphism then  $A'$  has the same form as  $A$  if and only if  $\varphi$  belongs to  $\Phi_\pi$ , where  $\pi \in S_{2n}^b = \{\pi = (\pi(1), \dots, \pi(2n)) \in S_{2n} : \pi(i) \in \{\pi(i-1) \pm 1\}, i = 1, 2, \dots, 2n\}$ .*

*Proof.* Using given above formula (5.5) for  $A'$  and the condition  $\det(A) \neq 0$  one can see that it has form as  $A$  iff  $\pi(i) \in \{\pi(i-1) \pm 1\}$ ,  $i = 1, 2, \dots, 2n$ .  $\square$

Properties of the matrix  $A$  can be uniquely defined by properties of its non-zero blocks. So if we consider  $n = 1$  then for  $\det(A) \neq 0$  we have two classes of isomorphisms:

$$\Phi_{12} = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} : \alpha\delta \neq 0 \right\} \quad \text{with} \quad A' = \begin{pmatrix} a\alpha & b\frac{\alpha^2}{\delta} \\ c\frac{\delta^2}{\alpha} & d\delta \end{pmatrix}.$$

$$\Phi_{21} = \left\{ \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix} : \beta\gamma \neq 0 \right\} \quad \text{with} \quad A' = \begin{pmatrix} d\beta & c\frac{\beta^2}{\gamma} \\ b\frac{\gamma^2}{\beta} & a\gamma \end{pmatrix}.$$

It is easy to check the following embedding:

$$\Phi_{12}\Phi_{21} \subset \Phi_{21}, \quad \Phi_{21}\Phi_{12} \subset \Phi_{21};$$

$$\Phi_{21}\Phi_{21} \subset \Phi_{12}, \quad \Phi_{12} \text{ is a group.}$$

Adding the symmetric property to the matrices  $A$  and  $A'$  we get the following classes of isomorphisms:

$$\Phi_{12}^s = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} : \alpha\delta \neq 0, ad - b^2 \neq 0, b\alpha^3 = b\delta^3 \right\},$$

$$\Phi_{21}^s = \left\{ \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix} : \beta\gamma \neq 0, ad - b^2 \neq 0, b\beta^3 = b\gamma^3 \right\}.$$

## APPENDIX

The following program written in Mathematica permits to check the existence (or non existence) of an isomorphism between two evolution algebras of dimension  $n$ . It is based on the star product of two evolution matrices,  $A*B$ , see [4, page 31] and the computation of Gröbner bases. In particular, one can check again that the algebras  $E_i, i = 1, \dots, 6$  (see Theorem 4.1) are pairwise non isomorphic.

```
StarProduct[A_List, B_List] := Module[{icont, jcont, kcont,
AEB, ndim, Indices}, ndim = Dimensions[A][[1]];
Indices = {};
Do[Indices = Join[Indices, {{icont, jcont}}];
, {icont, 1, ndim}, {jcont, icont + 1, ndim}];
AEB = Table[Table[aux[icont, jcont], {icont, 1, ndim}],
{jcont, 1, (ndim^2 - ndim)/2}];
```

```

Do[AEB[[icont, kcont]] =
  A[[Indices[[icont]][[1]], kcont]]*
  B[[Indices[[icont]][[2]], kcont]];
  ,{icont, 1, Length[Indices]}, {kcont, 1, ndim}];
  Return[AEB];]
SystemEquations[P_List, Q_List] :=Module[{FirstEquation,
SecondEquation, ThirdEquation, A, ndim, Result},
  ndim = Dimensions[P][[1]];
  A = Table[Table[aux[icont, jcont], {jcont, 1, ndim}],
    {icont, 1, ndim}];
  FirstEquation = (A*A).Q - P.A;
  SecondEquation = StarProduct[Transpose[A], Transpose[A]].Q;
  ThirdEquation = {Det[A]*Y - 1};
  Result = Join[Flatten[FirstEquation],
    Flatten[SecondEquation], ThirdEquation]; Return[Result];]
IsoEvolAlgebrasQ[A1_, A2_] := Module[{Equations, BGrobner},
  Equations = SystemEquations[A1, A2];
  BGrobner = GroebnerBasis[Equations, Variables[Equations]];
  (* Print temporal *)
  Print[BGrobner];
  If[BGrobner == {1},
    Print["Evolution algebras are NOT isomorphic"];
    Print["Evolution algebras are isomorphic"]; ];]

```

**Example 5.3.** *We check that the evolution algebras  $E_5$  and  $E_6$  are not isomorphic.*

```

IsoEvolAlgebrasQ[{{1, a2}, {a3, 1}}, {{0, 1}, {1, a4}}]
{1}
Evolution algebras are NOT isomorphic

```

#### ACKNOWLEDGEMENTS

The first and second authors were supported by Ministerio de Ciencia e Innovación (European FEDER support included), grant MTM2009-14464-C02, and by Xunta de Galicia, grant Incite09 207 215 PR. The third author was partially supported by the Grant NATO-Reintegration ref. CBP.EAP.RIG. 983169. The fourth author thanks to the Department of Algebra, University of Santiago de Compostela, Spain, for providing financial support of his visit to the Department.



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